Set Theory

"A *set* may be viewed as any well-defined collection of objects;
the objects are called the *elements* or *members* of the set."

"The concept of a set appears in all branches of mathematics.
This concept formalizes the idea of grouping objects together and viewing them as a single entity."

– Seymour Lipschutz, *Set Theory and Related Topics*[1]

TERMS & CONCEPTS

Set Theory
Mathematical set theory[2]
- Naive
- Axiomatic

Pitch-class set theory

Set
Members, or elements
- The order of the members does not matter
- Repeated members are ignored

Set notation
- Tabular form: a listing of a set's members
  ex. $A = \{a, e, i, o, u\}$
- Property method
  ex. $\{x \mid x \text{ is a vowel}\}$
- Sets are denoted by capital letters
  ex. *A, B, X, Y, etc.*

Finite vs. infinite sets
Cardinality: $|A|$
The number of distinct members in a set
Singleton set

***U* & ∅**
Universal set: ***U***
ex. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
Empty set: {} or ∅

Set Relations
Equality: $A = B$
Negation: $A \neq B$
Inclusion: x is a member of A: $x \in A$
Exclusion: x is not a member of A: $x \notin A$

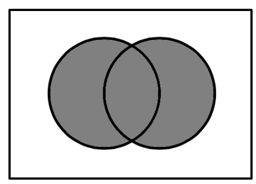
Subsets
Subset relation: $A \subset B$
Every member of A is a member of B
Superset relation: $A \supset B$
A contains B, thus $B \subset A$
Subset negation: $A \not\subset B$
Proper subset
A is a subset of B, but $A \neq B$

Power set: $P(A)$
All subsets of a given set.
If A has n members, then P(A) has 2^n *members.*

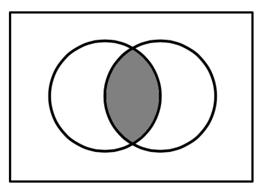
Set Operations
Union: $A \cup B$
The joining of two sets.
The set of all members that belong to A or B.

Intersection: $A \cap B$
The common members of A and B.
The set of all members that belong to both A and B.

Venn Diagrams
U *is denoted by a rectangle. Sets are denoted by overlapping and non-overlapping circles.*



$A \cup B$



$A \cap B$

Other Set Relations and Operations
Disjoint sets: $A \cap B = \varnothing$
Two sets that have no common members
Complement: A'
The set of all members that belong to ***U****, but not to A*
Cartesian product: $A \times B$
The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

[1] S. Lipschutz, *Schaum's Outline of Theory and Problems of Set Theory and Related Topics*, 2nd ed. (New York: McGraw Hill, 1998).
[2] For mathematical definitions, see *Mathematical Terms & Concepts* on the course website.

QUOTABLE

> "**Zermelo–Fraenkel set theory**…is a rigorous theory, based on a precise set of axioms. However, it is possible to develop the theory of sets considerably without any knowledge of those axioms…. The concept of a 'set of objects' is a very intuitive one, and, with care, considerable, sound progress may be made on the basis of this intuition alone."
>
> – Keith Devlin, *The Joy of Sets*[3]

> "A **set** is any well-defined collection of objects called the **elements** or **members** of the set…. One way of describing a set that has a finite number of elements is by **listing** the elements of the set between braces…. The order in which the elements of a set are listed is not important…. Moreover, repeated elements in the list of a set can be ignored."
>
> "…every set is a subset of itself and the empty set (∅) is a subset of every set."
>
> "Almost all mathematical objects are first of all sets…. Thus set theory is, in a sense, the foundation on which virtually all of mathematics is constructed."
>
> "The set of everything, it turns out, cannot be considered defined without destroying the logical structure of mathematics. To avoid this and other problems…we will assume that for each discussion there is a 'universal set' ***U*** (which may vary with the discussion) containing all objects for which our discussion is meaningful."
>
> – B. Kolman and R. Busby, *Discrete Mathematical Structures*[4]

EXAMPLE 1

Consider the following *equivalent* representations of the set {1, 2, 3}:

{1, 2, 3}, {2, 3, 1}, {3, 2, 1}, {1, 2, 1, 3}, {2, 3, 2, 3, 1}, etc.

EXAMPLE 2

Consider the following sets, relations, and operations:

A = {a, b, c} B = {d, e, f, g} C = {f, g} D = {d, c, b, a} E = {g, f, g, f}

a. $A \cup B$ = {a, b, c, d, e, f, g}	**e**. $A \cap B = \varnothing$	**i**. $C \subset B$
b. $A \cup C$ = {a, b, c, f, g}	**f**. $A \cap D$ = {a, b, c}	**j**. $A \not\supset B$
c. $A \cup D$ = {a, b, c, d}	**g**. $A \subset D$	**k**. $C = E$
d. $B \cap C$ = {f, g}	**h**. $D \supset A$	**l**. $A \neq B$

EXAMPLE 3: PITCH-CLASS SETS

Consider the following pc sets, relations, and operations:

A = {C, D, E, F, G, A, B}

W = {C, E, G} X = {F, A, C} Y = {G, B, D} Z = {G, B, D, F}

a. $W \cup X \cup Y = A$	**e**. $Y \cap Z$ = {G, B, D}
b. $W \cup Y$ = {C, E, G, B, D}	**f**. $Y \subset Z$, and $Z \supset Y$
c. $W \cap Y$ = {G}	**g**. $X \cap Y = \varnothing$
d. $W \cap X$ = {C}	**h**. A' = {F♯, G♯, A♯, C♯, D♯}

Notice that set A is equivalent to a diatonic collection on C and pc sets W, X and Y are equivalent to the tonic (I), subdominant (IV), and dominant (V) chords in the key of C major, respectively. Similarly, Z is equivalent to the dominant seventh chord (V^7). The union (∪) operation models the joining two chords: e.g., $W \cup Y$ forms a Cmaj^9 chord. Intersection (∩) models common-tone relationships, and the complement of A (i.e., the white keys on the piano) is the black keys. Etc.

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[3] Keith Devlin, *The Joy of Sets: Fundamentals of Contemporary Set Theory*, 2nd ed. (New York: Springer, 1994).
[4] B. Kolman and R. C. Busby, *Discrete Mathematical Structures for Computer Science* (Englewood Cliffs, NJ: Prentice-Hall, 1984).